

LQG vertex with finite Immirzi parameter

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We extend the definition of the “flipped” loop-quantum-gravity vertex to the case of a finite Immirzi parameter γ . We cover both the euclidean and lorentzian cases. We show that the resulting dynamics is defined on a Hilbert space isomorphic to the one of loop quantum gravity, and that the area operator has the same discrete spectrum as in loop quantum gravity. This includes the correct dependence on γ , and, remarkably, holds in the lorentzian case as well. The *ad hoc* flip of the symplectic structure that was required to derive the flipped vertex is not anymore required for finite γ . These results establish a bridge between canonical loop quantum gravity and the spinfoam formalism in four dimensions.

I. INTRODUCTION

The Barrett-Crane (BC) vertex, which provides a tentative definition of the quantum-gravity dynamics, has been extensively investigated during the past years [1]; its amplitude is essentially given by a Wigner $10j$ symbol. A different vertex has been recently introduced in [2, 3]; its amplitude essentially given by the square of an $SU(2)$ Wigner $15j$ symbol. There are indications that this new vertex could ameliorate the properties of the BC model. First, it appears to correct an over-imposition of the constraints that was remarked in the derivation of the BC vertex. Second, it does not appear to freeze the angular degrees of freedom of the gravitational field (that is, $g_{ab}(x)$ for $a \neq b$) as it has been argued the BC model might do [4]. Third, preliminary numerical investigations appear to be consistent with the expectation that geometry wave packets are propagated by this new vertex in a way consistent with euclidean general relativity (GR) [5]. And finally, its kinematics matches exactly the one of the canonical quantization of GR, as given by loop quantum gravity (LQG) [6]. The vertex was defined in [2, 3] only for the euclidean case, and in the absence of an Immirzi parameter γ .

A key step to extend the definition of this new vertex was taken in [7], where a lorentzian version of the vertex amplitude is constructed, still without γ . Here we extend the construction of the vertex to the general case of finite γ , both for the euclidean and the lorentzian sectors.

As long emphasized by Sergei Alexandrov [8], the key technical problem is how to impose the second class quantum constraints in a covariant way (see [9]). These constraints are solved in [2, 3] using a master-constraint-like [10] technique. In [11], it was shown that these constraints can equivalently be solved using a different technique, based on coherent states, yielding the same result. This derivation reinforces the credibility of the approach, and opens a direct connection to the semiclassical limit.

In the same paper [11], on the other hand, it was also pointed out that considering a different class of coherent states leads to a variant of the model. This variant has been extensively explored in [12], and extended to the case of finite $\gamma > 1$. The original model of [2, 3] and

the variant pointed out in [11] appear as limiting $\gamma \rightarrow 0$ and $\gamma \rightarrow \infty$ cases, respectively. All these models ([2, 3, 11, 12],) are defined by the same vertex, namely the square of the $SU(2)$ Wigner $15j$ symbol; they differ for the class of boundary states considered and their measure in the spinfoam sum. In [13], on the other hand, it was observed that the use of coherent states may not truly constraint the physical state space of the theory when the constraints are not entirely second class, and this happens in the limit case $\gamma \rightarrow \infty$. Therefore, while the coherent state technique introduced in [11] appears to work well in the $\gamma \rightarrow 0$ case, its straightforward extension to large γ yields a state space larger than the physical state space of LQG and -one might argue- larger than the proper quantum state space of gravity. Furthermore, the spectrum of the geometrical operators in this formulation turns out to be quite different from the standard one of loop quantum gravity [14]. Here, thus, we reconsider the finite γ case, but we solve the constraints using the same master constraint technique as in [2, 3]. We leave the understanding of our results in terms of coherent states for future developments.

We find a model with a number of interesting properties. First, the second class constraints do reduce the dimension of the physical state space as one wants. Second, for all values of γ the state space precisely matches the one of LQG (on a fixed graph). This is particularly interesting in the case of the lorentzian theory, where such a match is traditionally more problematic. Third, the spectrum of the area operator turns out to be discrete, and to be the same as in LQG, including the correct dependence on the Immirzi parameter γ . What is of particular interest is that this is true in the lorentzian case as well, in spite of the fact that the unitary representations of the Lorentz group are labelled also by a continuous parameter. This provides a solution to a long-standing controversy: the area spectrum is discrete in LQG while it appears to be continuous in the spinfoam framework. The solution is that the area spectrum is continuous in spinfoams at the kinematical level, but it turns out to become discrete after proper implementation of the (second class) constraints. Finally, the *ad hoc* “flip” of the symplectic structure used to first derive the vertex in [2, 3]

is not required in the finite γ case.

All these developments rely on two basic ideas. The first, championed by Giorgio Immirzi [15], is to (“loop”) quantize GR by first discretizing it on a Regge-like triangulation, with appropriately chosen variables. The second is to treat the simplicity constraints by first imposing them properly in a fixed $SO(4)$ (or $SO(3,1)$) gauge, and then projecting on the gauge invariant spaces. The implementation of these ideas is discussed in detail in [2, 3]. Here, we briefly describe in Section II the discrete theory introduced in [2, 3] (euclidean case) and in [7] (lorentzian case), in order to make this paper self contained. We define the euclidean theory in Section III then the lorentzian theory in Section IV. Finally, Section 5 is devoted to the spectrum of the area operator.

We work on a fixed triangulation. It was shown in [16] that any spinfoam model [17], as the one we define here, admits a group field theory (GFT) [18] formulation, which is triangulation independent. A GFT yielding the new vertex amplitude is indeed already considered in [12]. We leave the complete construction of the background independent GFT corresponding to the model defined here to future developments.

II. DISCRETE MODEL FOR FINITE γ

A. Classical theory

We quantize GR by first discretizing the theory on a Regge geometry. Introduce a simplicial decomposition Δ of space-time, consisting of 4-simplices, tetrahedra, and triangles. These are dual respectively to vertices v , edges e , and faces f in the dual 2-complex. Geometry is assumed to be flat on each 4-simplex; curvature is concentrated on the “bones” f , and is coded in the holonomy around the “link” of each f . The variables to describe this geometry are chosen as follows. (See [13] for a precise and detailed definition.) $e(v)_a^I$ is a tetrad one-form in a cartesian coordinate patch covering the simplex v . Here a, b are 4d tangent indices and I, J are 4d internal indices. $e(t)_a^I$ is a tetrad one-form in a cartesian coordinate patch covering the tetrahedron t . The matrix $(V_{vt})^I_J$ is defined by $(V_{vt})^I_J e(v)_a^J = e(t)_a^I$ when t bounds v and in a common coordinate patch. It belongs to the group $G = SO(4)$ in the euclidean case and $G = SO(3,1)$ in the lorentzian case.

For each triangle f in t , define

$$B_f(t)^{IJ} := \int_f \star(e(t)^I \wedge e(t)^J). \quad (1)$$

where \star stand for the Hodge dual in the internal indices. $B_f(t)$ can be seen as element in the algebra $\mathfrak{g} = so(4)$ in the euclidean case and $\mathfrak{g} = so(3,1)$ in the lorentzian case. For each triangle f and each pair of tetrahedra t, t' in the link of f , define

$$U_f(t, t') := V_{tv_1} V_{v_1 t_1} V_{t_1 v_2} \cdots V_{v_n t'}, \quad (2)$$

where the product is around the link in the clock-wise direction from t' to t .

If we choose $B_f(t)$ as independent variable instead of the tetrads, the constraints on $B_f(t)$ can be stated as follows.

- $\forall f$ and $t, t' \in \text{Link}(f)$,

$$U_f(t, t') B_f(t') = B_f(t) U_f(t, t'); \quad (3)$$

- (closure) $\forall t$

$$\sum_{f \in t} B_f(t) = 0; \quad (4)$$

- (diagonal simplicity constraint) $\forall f$

$$C_{ff} := \star B_f(t) \cdot B_f(t) \approx 0; \quad (5)$$

- (off-diagonal simplicity constraint) $\forall f, f' \in t$

$$C_{ff'} := \star B_f(t) \cdot B_{f'}(t) \approx 0; \quad (6)$$

- (dynamical simplicity constraint) $\forall f, f' \in v$ not in the same t

$$\star B_f(v) \cdot B_{f'}(v) \approx \pm 12V(v). \quad (7)$$

The dot stands for the scalar product in the algebra. As noted in [3, 11], (7) is automatically satisfied when the rest of the constraints are satisfied, due to the choice of variables; we can therefore forget about it. When constructing the quantum theory, the above constraints are incorporated as follows. (3) is assumed to hold prior to varying the action. (4) is automatically implemented by the dynamics in quantum theory, because (4) generates the internal gauge, and the vertex amplitude turns out to project on the gauge invariant subspace. Finally, (5) and (6) must be separately imposed on the state space. The complication is that they are not first class.

The above is the usual formulation of the simplicity constraints. As written, however, the two constraints (5) and (6) have two sectors of solutions, one in which $B = \star e \wedge e$, and one in which $B = e \wedge e$. For finite, non-trivial Immirzi parameter, both sectors in fact yield GR, but the value of the Newton constant and Immirzi parameter are different in each sector. In the $B = \star e \wedge e$ sector, the action (9) (see below) becomes the Holst formulation of GR [19] with Newton constant G and Immirzi parameter γ . In the $B = e \wedge e$ sector, one *also* obtains the Holst formulation of GR, but this time with Newton constants $G\gamma$, and Immirzi parameter s/γ , where the signature s is $+1$ in the euclidean theory and -1 in the lorentzian theory. In order to select a single sector, we reformulate the simplicity constraints in such a way that these two sectors are distinguished. For this purpose, we replace (6) with

- For each tetrahedron t , there exists an internal vector n_I such that for all $f \in t$

$$C_f^J := n_I (*B_f(t))^{IJ} \approx 0. \quad (8)$$

This condition is stronger than (6) since it selects only the desired $B = *e \wedge e$ sector. Geometrically, n_I represents of course the normal one-form to the tetrahedron t . This reformulation of the constraint (6) is central to the new models [2, 3, 7, 11, 12]. The vector n_I already played a central role in the covariant loop quantum gravity approach, which is known to be closely related to 4d spinfoam models [8, 9].

The classical discrete action is [3]

$$S = -\frac{1}{2\kappa} \sum_{f \in \text{int}\Delta} \text{tr} \left[B_f(t) U_f(t) + \frac{1}{\gamma} *B_f(t) U_f(t) \right] - \frac{1}{2\kappa} \sum_{f \in \partial\Delta} \text{tr} \left[B_f(t) U_f(t, t') + \frac{1}{\gamma} *B_f(t) U_f(t, t') \right] \quad (9)$$

where $U_f(t) := U_f(t, t)$ is the holonomy around the full link, starting at t , we have set $\kappa = 8\pi G$, and $\text{int}\Delta$ and $\partial\Delta$ are the interior and the boundary of Δ . S is a discretization of the continuous action

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} \left[B_{IJ} \wedge F^{IJ} + \frac{1}{\gamma} (*B)_{IJ} \wedge F^{IJ} \right] + \frac{1}{2\kappa} \int_{\partial\mathcal{M}} \left[B_{IJ} \wedge F^{IJ} + \frac{1}{\gamma} (*B)_{IJ} \wedge F^{IJ} \right]$$

which becomes the Holst action (see [6, 19]) on substituting in $B = *e \wedge e$.

From this we can read off the boundary variables as $B_f(t) \in \mathfrak{g}$, $U_f(t, t') \in G$. The variable conjugate to $U_f(t, t')$ is

$$J_f(t) = \frac{1}{\kappa} \left(B_f(t) + \frac{1}{\gamma} *B_f(t) \right) \quad (10)$$

(on the determination of the normalization factor see [13].) That is, the matrix elements $J_f(t)^{IJ}$ have as their Hamiltonian vector fields the right invariant vector fields on the group $U_f(t, t')$. Inverting this equation gives

$$B_f(t) := \left(\frac{\kappa\gamma^2}{\gamma^2 - s} \right) \left(J_f(t) - \frac{1}{\gamma} *J_f(t) \right), \quad (11)$$

where s is the signature, namely $s = 1$ for $G = SO(4)$ and $s = -1$ for $G = SO(3, 1)$. For the cases $\gamma \ll 1$ and $\gamma \gg 1$, this reduces to

$$\gamma \ll 1 \rightsquigarrow B_f = s\kappa\gamma *J_f, \quad \gamma \gg 1 \rightsquigarrow B_f = \kappa J_f.$$

corresponding respectively to the flipped and non flipped Poisson structures of G . The constraints (5) and (8) can be easily expressed in terms of the new variables J_f

$$C_{ff} := *J_f \cdot J_f \left(1 + s \frac{1}{\gamma^2} \right) - s \frac{2}{\gamma} J_f \cdot J_f \approx 0, \quad (12)$$

$$C_f^J := n_I \left((*J_f)^{IJ} - s \frac{1}{\gamma} J_f^{IJ} \right) \approx 0 \quad (13)$$

(see [20] and [13]) where we have assumed γ finite and $\neq 0, 1$.

The closure for the B_f is equivalent to the closure for the J_f which, as noted above, will be imposed automatically by the dynamics. In order to proceed, let us fix $n_I = \delta_I^0$. The general case will be recovered by gauge invariance. In the lorentzian case this choice restricts all tetrahedra to be spacelike (it is not clear to us if a non-timelike choice for n_I is viable: see [21]). With this choice, the constraint (13) becomes:

$$C_f^j = \frac{1}{2} \epsilon^j_{kl} J_f^{kl} - s \frac{1}{\gamma} J_f^{0j} = L_f^j - s \frac{1}{\gamma} K_f^j \approx 0, \quad (14)$$

where $\epsilon^j_{kl} := \epsilon^{0j}_{kl}$, $L_f^j := \frac{1}{2} \epsilon^j_{kl} J_f^{kl}$ are the generators of the $SO(3)$ subgroup that leaves n_I invariant, and $K_f^j := J_f^{0j}$ are the generators of the corresponding boosts.

We take (12, 14) as our basic set of constraints. So far, we have simply formulated a discretization of GR.

B. Quantum kinematics

From the discrete boundary variables and their symplectic structure, we can write the Hilbert space associated with a boundary or a 3-slice Σ . To do this, it is simpler to switch to the dual, 2-complex picture, Δ^* . Let Γ be the graph forming the boundary of Δ^* . The boundary Hilbert space is then

$$\mathcal{H} = L^2(G^{\times L}), \quad (15)$$

where L is the number of links in Γ , namely the number of boundary faces f . As is standard, we replace the groups G with their universal coverings in the quantum theory. That is, from now on $G = Spin(4)$ in the euclidean case and $G = SL(2, \mathbb{C})$ in the lorentzian case.

Let us concentrate on the single $L^2(G)$ component of \mathcal{H} associated to a single boundary face f . The face f is dual to the link $l \in \Gamma$ bounding f . The orientation of the triangulation selects one of the two boundary tetrahedra separated by f : call it t . (t is dual to the node which is the *source* of the link l). For simplicity of notation, let us drop the subscript f and the dependence (t) all over, and rewrite $B_f(t)^{IJ}$, $J_f(t)^{IJ}$, C_f^i simply as B^{IJ} , J^{IJ} , C^i . Let \hat{J}^{IJ} denote the right-invariant vector fields, determined by the basis J^{IJ} of \mathfrak{g} . From (11), the variable $B = B_f(t)$ associated to the boundary face f (and the t determined by the orientation mentioned above) is then quantized as

$$\hat{B} := \left(\frac{\kappa\gamma^2}{\gamma^2 - s} \right) \left(\hat{J} - \frac{1}{\gamma} * \hat{J} \right). \quad (16)$$

Equation (3) implies then that the quantities associated to the other tetrahedron t bounding f are given by the corresponding *left*-invariant vector fields [2, 3].

Next we impose the constraints (12) and (14) in the quantum theory. The constraint (12) commutes with the

others and can be imposed directly as a strong operator equation. It reads:

$$C_2 \left(1 + \frac{s}{\gamma^2} \right) - \frac{2s}{\gamma} C_1 \approx 0 \quad (17)$$

where C_1 and C_2 are the Casimir and pseudo-Casimir operators of \mathfrak{g} .

$$C_1 = J \cdot J = 2(L^2 + sK^2), \quad (18)$$

$$C_2 = {}^*J \cdot J = 4sL \cdot K. \quad (19)$$

L^2 is the Casimir of the $SU(2)$ subgroup that leaves n_I fixed. The constraints (14) on the other hand do not close as a Poisson algebra and their imposition in the quantum theory is more subtle. We follow the strategy of [10] and replace the set of these constraints with the single “master” constraint

$$M_f := \sum_i (C^i)^2 = \sum_i \left(L^i - \frac{s}{\gamma} K^i \right)^2 \approx 0. \quad (20)$$

which is of course equivalent to the system (14) in the classical theory. In terms of the Casimir operators, $M_f \approx 0$ gives:

$$L^2 \left(1 - \frac{s}{\gamma} \right) + \frac{s}{2\gamma^2} C_1 - \frac{1}{2\gamma} C_2 \approx 0, \quad (21)$$

While equation (17) was already known [20], this last relation was noticed only in [13], as far as we know. We can finally use (17) to simplify this last equation obtaining

$$C_2 = 4\gamma L^2. \quad (22)$$

The solutions to (17)-(22) will depend on the particular group G . In the next sections we analyze separately the cases for $G = Spin(4)$ and $G = SL(2, \mathbb{C})$.

III. EUCLIDEAN THEORY

The unitary representation of $Spin(4)$ are labelled by the two half integers (j^+, j^-) . With the usual ordering (and with our normalizations) the two Casimirs (18) and (19) have the values

$$C_1 = 4j^+(j^+ + 1) + 4j^-(j^- + 1), \quad (23)$$

$$C_2 = 4j^+(j^+ + 1) - 4j^-(j^- + 1) \quad (24)$$

in the representation (j^+, j^-) . The constraint (17) fixes the ratio between these two Casimirs in term of the Immirzi parameter γ . Choosing a suitable ordering (or, equivalently, up to \hbar corrections), solutions are given by

$$(j^+)^2 = \left(\frac{\gamma + 1}{\gamma - 1} \right)^2 (j^-)^2. \quad (25)$$

An ordering ambiguity is always present in quantum theory. For example, different orderings of the Casimir can

yields spectra $j(j+1)$ or j^2 or $(j+1/2)^2$, or anything differing from these by a linear or constant shift. In the spinfoam context, this ambiguity can be related to ambiguities in the path integral measure. In the present context, the natural ordering in order to find solutions to the simplicity constraints seems to favor the spectrum j^2 (or $(j+1/2)^2$) for the $SU(2)$ Casimir operator instead of the usual $j(j+1)$. This would lead to an area spectrum with a constant spacing of the type j (or $(j+1/2)$).

In (25), we distinguish the two cases: if $\gamma > 0$ then $j^+ > j^-$; while if $\gamma < 0$ then $j^+ < j^-$. Let us restrict to the case $\gamma > 0$. Notice that this equation imposes a quantization condition over γ as the labels j^\pm are half-integers. This was pointed out in [12]. Inserting this into the second simplicity constraint (22) (and, again, allowing for \hbar corrections) constrains the quantum number k associated to the $SU(2)$ Casimir L^2 to

$$k^2 = \left(\frac{2j^-}{1-\gamma} \right)^2 = \left(\frac{2j^+}{1+\gamma} \right)^2, \quad (26)$$

The solutions are substantially different for $\gamma < 1$ and for $\gamma > 1$ (the value $\gamma = 1$ is the natural turning point in the euclidean setting since it corresponds to a pure self-dual connection):

$$k = \begin{cases} j^+ + j^- & 0 < \gamma < 1, \\ j^+ - j^- & \gamma > 1. \end{cases} \quad (27)$$

That is, for $\gamma < 1$, the constraint selects the highest irreducible in the decomposition of $\mathcal{H}_{(j^+, j^-)}$ when viewed as the carrying space of a reducible representation under the action of the $SU(2)$ subgroup: $\mathcal{H}_{(j^+, j^-)} = \mathcal{H}_{|j^+ - j^-|} \oplus \dots \oplus \mathcal{H}_{j^+ + j^-}$. For $\gamma > 1$ the lowest irreducible is selected instead.

A prescription similar to ours for the $\gamma < 1$ case already appeared earlier in [12], where the authors impose $k = j_+ + j_-$ for $\gamma > 1$ for what they call the “topological” sector. However the term “topological” is not accurate; for, what is meant by this is the $B = e \wedge e$ sector and, as already noted earlier, this sector does not correspond to a topological sector of gravity but truly to general relativity with effective Immirzi parameter $1/\gamma$. The model in [12] is simply related to our $\gamma < 1$ model by $\gamma \mapsto 1/\gamma$. Finally, unlike [12], we find solutions to the simplicity constraints in both $\gamma > 1$ and $\gamma < 1$ cases.

The component of $\mathcal{H} = L^2(Spin(4)^{\times L})$ associated to each face f decomposes as

$$L^2(Spin(4)) = \bigoplus_{j^+ j^-} \overline{H_{j^+ j^-}} \otimes H_{j^+ j^-}. \quad (28)$$

The diagonal simplicity constraint restricts the direct sum to spins satisfying (25). The off-diagonal simplicity constraints select the $SU(2)$ irreducible with the spin determined by (27) in each of the two factors. We call this constrained subspace \mathcal{H}_f .

\mathcal{H}_f can be naturally identified with $L^2(SU(2))$. The projection

$$\pi : L^2(Spin(4)) \longrightarrow L^2(SU(2)) \sim \mathcal{H}_f \quad (29)$$

can be written explicitly as follows. A basis in $L^2(Spin(4))$ is formed by the matrix elements $D_{q^+q^-, q'^+q'^-}^{(j^+, j^-)}(g)$ of the irreducible representations. Here $g \in Spin(4)$, and the indices q^\pm label a basis in the representation j^\pm . Then

$$\pi : D_{q^+q^-, q'^+q'^-}^{(j^+, j^-)}(g) \mapsto D_{q^+q^-, q'^+q'^-}^{(j^+, j^-)}(u) c_m^{q^+q^-} c_{m'}^{q'^+q'^-}.$$

where $u \in SU(2)$ and the $c_m^{q^+q^-}$ are the Clebsch-Gordan coefficients that gives the embedding of the lowest (resp. highest) $SU(2)$ irreducible (where the m index lives) into the representation (j^+, j^-) . This construction defines also an embedding from the $SU(2)$ spin networks to the $Spin(4)$ spin networks on Γ . This is defined by the embedding of $L^2(SU(2)^{\times L})$ into $L^2(Spin(4)^{\times L})$ defined by the inclusion $L^2(SU(2)) \sim \mathcal{H}_f \subset L^2(Spin(4))$ followed by the group averaging over $Spin(4)$ at every node, as determined by the constraint (4) (which, we recall, is implemented by the dynamics).

Let us see how this construction affects the intertwiner spaces. We decompose the Hilbert space associated with each face into representations. The simplicity and cross-simplicity constraints, as discussed above, are then imposed on each of these representations. Consider four links, colored with the representations $(j_1^+, j_1^-) \dots (j_4^+, j_4^-)$, satisfying (25), meeting at a given node e of Γ . (This is the dual picture of four faces bounding a given tetrahedron in the boundary of the triangulation). Consider the tensors product of the corresponding representation spaces $\mathcal{H}_e := \mathcal{H}_{(j_1^+, j_1^-)} \otimes \dots \otimes \mathcal{H}_{(j_4^+, j_4^-)}$. Define the constraint $C_e := \sum_i M_{f_i}$. Imposing $C_e = 0$ strongly on the states in \mathcal{H}_0 selects in each link the lowest (resp. highest) $SU(2)$ irreducible. Group averaging over $Spin(4)$ defines then the physical intertwiner space for the node e . The projection from the $Spin(4)$ to the $SU(2)$ intertwiner spaces is then given by:

$$\pi : Inv_{Spin(4)}(\mathcal{H}_e) \rightarrow Inv_{SU(2)}(\mathcal{H}_{j_1^+ \pm j_1^-} \otimes \dots \otimes \mathcal{H}_{j_4^+ \pm j_4^-})$$

$$C_{(q_1^+ q_1^-) \dots (q_4^+ q_4^-)}^{(i_e^+, i_e^-)} \mapsto C_{(q_1^+ q_1^-) \dots (q_4^+ q_4^-)}^{(i_e^+, i_e^-)} \bigotimes_{i=1}^4 c_{m_i}^{q_i^+ q_i^-}.$$

Here $C_{(q_1^+ q_1^-) \dots (q_4^+ q_4^-)}^{(i_e^+, i_e^-)}$ is the normalized intertwiner defined by a virtual link carrying the (i_e^+, i_e^-) representation. The corresponding embedding can be written in the form:

$$f : Inv_{SU(2)}(\mathcal{H}_{k_1} \otimes \dots \otimes \mathcal{H}_{k_4}) \rightarrow Inv_{Spin(4)}(\mathcal{H}_e)$$

$$i^{m_1 \dots m_4} \mapsto \int_{Spin(4)} dg i^{m_1 \dots m_4}$$

$$\times \bigotimes_{i=1}^4 D_{q_i^+ q_i^-, q_i'^+ q_i'^-}^{\frac{(1+\gamma)k_i}{2}, \frac{|1-\gamma|k_i}{2}}(g) c_{m_i}^{q_i^+ q_i'^-}. \quad (30)$$

We are now ready to define the vertex. For the details of the derivation, see [24] and [7]. Following [2, 3, 7], the

amplitude of a single vertex bounded by ten $SU(2)$ spins $j_{ab}, a, b = 1, \dots, 5$ and five $SU(2)$ intertwiners i_a is given by

$$A(j_{ab}, i_a) = \sum_{i_a^+ i_a^-} 15j\left(\frac{(1+\gamma)j_{ab}}{2}; i_a^+\right) 15j\left(\frac{|1-\gamma|j_{ab}}{2}; i_a^-\right)$$

$$\bigotimes_a f_{i_a^+ i_a^-}^{i_a}(j_{ab}) \quad (31)$$

where the $15j$ are the standard $SU(2)$ Wigner symbols, and

$$f_{i^+ i^-}^i := i^{m_1 \dots m_4} C_{(q_1^+ q_1^-) \dots (q_4^+ q_4^-)}^{i^+ i^-} \bigotimes_{i=1 \dots 4} c_{m_i}^{q_i^+ q_i^-}. \quad (32)$$

The partition function for an arbitrary triangulation, is given by gluing these amplitudes together with suitable edge and face amplitudes. It can be written as:

$$Z = \sum_{j_f, i_e} \prod_f d_f \prod_v A(j_f, i_e), \quad (33)$$

where

$$d_f := (|1 - \gamma|j_f + 1)((1 + \gamma)j_f + 1). \quad (34)$$

IV. LORENTZIAN THEORY

The unitary representations in the principal series are labelled by (n, ρ) , where n is a positive integer and ρ real [22, 23]. The Casimir operators for the representation (n, ρ) , are given by

$$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4), \quad (35)$$

$$C_2 = n\rho. \quad (36)$$

Up to ordering ambiguities, equation (17) reads now

$$n\rho \left(\gamma - \frac{1}{\gamma} \right) = \rho^2 - n^2. \quad (37)$$

Solutions are given by either $\rho = \gamma n$ or $\rho = -n/\gamma$. The existence of these two solutions reflects the two sectors mentioned earlier with Immirzi parameter γ and $-1/\gamma$. BF theory can not a priori distinguish between these two sectors (see e.g. [20]). However, in our framework, the second constraint (22) breaks this symmetry and select the first branch $\rho = \gamma n$. It further imposes that $k = n/2$, where k again labels the subspaces diagonalizing L^2 . Therefore the constraints select the lowest $SU(2)$ irreducible representation in the decomposition of $\mathcal{H}_{(n, \rho)} = \bigoplus_{k \geq n/2} \mathcal{H}_k$. This choice of the lowest weight corresponds to the usual notion of coherent states for the non-compact $SL(2, \mathbb{C})$ Lie group [25] (see also [26]). Notice that there is restriction on the value of γ as there was in the Euclidean case.

Notice also that the continuous label ρ becomes quantized, because n is discrete. It is because of this fact that

any continuous spectrum depending on ρ comes out effectively discrete on the subspace satisfying the simplicity constraints.

This construction defines the projection from the $SL(2, \mathbb{C})$ boundary Hilbert space to the $SU(2)$ space. For a single D matrix, this projection reads (see the [7]):

$$\begin{aligned} \pi : L^2(SL(2, \mathbb{C})) &\longrightarrow L^2(SU(2)) \\ D_{jqj'q'}^{n, \rho}(g) &\longmapsto D_{qq'}^{n/2}(u) \end{aligned} \quad (38)$$

This also defines an embedding from the $SU(2)$ Hilbert space to the $SL(2, \mathbb{C})$ space, given by inclusion followed by group averaging over the Lorentz group.

As before, in order to extend this result to the complete space \mathcal{H} we have to define the projection for the intertwiners. Consider four links meeting at a given node e of Γ , carrying representations $(n_1, \rho_1) \dots (n_4, \rho_4)$, satisfying the diagonal constraints. Consider the Hilbert space of tensors between these representations: $\mathcal{H}_e := \mathcal{H}_{(n_1, \rho_1)} \otimes \dots \otimes \mathcal{H}_{(n_4, \rho_4)}$. Construct the constraint $C_e := \sum_i M_{f_i}$. Imposing $C_e = 0$ strongly selects in each link the lowest $SU(2)$ along with the representations of the form $\rho = n\gamma$. The last step is group averaging over $SL(2, \mathbb{C})$ which defines the physical intertwiner space for this node. The projection is then given by:

$$\begin{aligned} \pi : Inv_{SL(2, \mathbb{C})}(\mathcal{H}_e) &\longrightarrow Inv_{SU(2)}\left(\mathcal{H}_{\frac{n_1}{2}} \otimes \dots \otimes \mathcal{H}_{\frac{n_4}{2}}\right), \\ C_{(j_1, q_1) \dots (j_4, q_4)}^{(n_e, \rho_e)} &\longmapsto C_{(\frac{n_1}{2}, q_1) \dots (\frac{n_4}{2}, q_4)}^{(n_e, \rho_e)}. \end{aligned} \quad (39)$$

The embedding is given by:

$$\begin{aligned} f : Inv_{SU(2)}(\mathcal{H}_{j_1} \otimes \dots \otimes \mathcal{H}_{j_4}) &\longrightarrow Inv_{SL(2, \mathbb{C})}(\mathcal{H}_e), \\ i^{m_1 \dots m_4} &\longmapsto \int_{SL(2, \mathbb{C})} dg \, i^{m_1 \dots m_4} \bigotimes_{i=1}^{i=4} D_{(j'_i, m'_i)(j_i, m_i)}^{(2j_i, 2j_i \gamma)}(g). \end{aligned}$$

The boundary space is once again just given by the $SU(2)$ spin networks.

We are now ready to define the vertex. As before, we obtain

$$\begin{aligned} A(j_{ab}, i_a) &= \sum_{n_a} \int d\rho_a (n_a^2 + \rho_a^2) \left(\bigotimes_a f_{n_a \rho_a}^{i_a}(j_{ab}) \right) \\ &15j_{SL(2, \mathbb{C})}((2j_{ab}, 2j_{ab}\gamma); (n_a, \rho_a)) \end{aligned} \quad (40)$$

where we are now using the 15j of $SL(2, \mathbb{C})$ and

$$f_{n\rho}^i := i^{m_1 \dots m_4} \bar{C}_{(j_1, m_1) \dots (j_4, m_4)}^{n\rho}, \quad (41)$$

where $j_1 \dots j_4$ are the representations meeting at the node. The final partition function, for an arbitrary triangulation, is given by gluing these amplitudes together with suitable edge and face amplitudes:

$$Z = \sum_{j_f, i_e} \prod_f (2j_f)^2 (1 + \gamma^2) \prod_v A(j_f, i_e). \quad (42)$$

V. AREA SPECTRA

There are two operators related to the area of a triangle dual to the face f .

$$A_4(f) := \frac{1}{2} (*B)^{IJ} (*B)_{IJ} \quad (43)$$

and its projected (gauge fixed) counterpart:

$$A_3(f) := \frac{1}{2} (*B)^{ij} (*B)_{ij} \quad (44)$$

Classically, these two quantities are equal due to the constraint (13). After quantization this will not hold anymore. This can be seen as follows. Since boosts do not commute, it is not possible in the quantum theory to physically implement a Lorentz frame exactly. Hence all spacelike vectors are affected by quantum fluctuation in the timelike directions. The relation between the two quantities above is given by

$$A_4 = A_3 + \left(\frac{\kappa \gamma^2}{\gamma^2 - s} \right)^2 s M_f. \quad (45)$$

Let us focus on A_3 , which is the standard canonical operator considered in a canonical quantization of GR. We can write

$$A_3 = \left(\frac{\kappa \gamma^2}{\gamma^2 - s} \right)^2 \left(\vec{K} - \frac{\vec{L}}{\gamma} \right)^2. \quad (46)$$

Using the constraints (17) and (22), we get with straightforward algebra

$$A_3 = \kappa^2 \gamma^2 L^2 \quad (47)$$

for both euclidean and lorentzian signatures. The spectrum is therefore

$$Area = \sqrt{A_3} = 8\pi \hbar G \gamma \sqrt{k(k+1)}. \quad (48)$$

which is *exactly* the spectrum of LQG. This spectrum can be compared with the continuous spectrum

$$Area \sim \frac{1}{2} \sqrt{4k(k+1) - n^2 + \rho^2 + 4}. \quad (49)$$

that was previously obtained in covariant LQG, before imposing the second class constraints (see [9]). Remarkably, imposing the simplicity constraints (17) and (22) reduces the continuous spectrum (49) to the exact discrete LQG spectrum (48).

Finally, we would like to point out that the ordering of the Casimir operators for $SU(2)$ and $SL(2, \mathbb{C})$ required to have meaningful simplicity constraints do not use the usual ordering but seems to select an area spectrum with a regular spacing such as j (or $j + 1/2$) instead of the standard $\sqrt{j(j+1)}$. This issue deserves further investigation.

VI. CONCLUSION

We have defined a spinfoam model for finite values of the Immirzi parameter γ , for the euclidean as well as for the lorentzian theory. In both cases, the boundary space turns out to be the same as in LQG, spanned by $SU(2)$ spin networks. The spectrum of the area operator too is the same as in LQG, both for the euclidean and the lorentzian sectors.

We leave the analysis of the model for future developments. Among the numerous issues we leave open is whether the vertex itself is finite in the lorentzian case, or whether it needs to be regulated; and whether it is possible to extend the spinfoam finiteness results [27] to the present model. It would be of particular interest to check

whether this model gives the correct graviton propagator [28].

One of the main results of this paper is to bring LQG and the spinfoam formalism much closer, in four dimensions. It would be of great interest if a direct relation between these two nonperturbative quantizations of GR could be completely established, as it was done in three dimensions by Perez and Noui [29]. For this, it would be necessary to write the hamiltonian constraint operator that generates the new vertex, that is, whose matrix elements are given by the new vertex amplitude.

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